Problem 1. LQ games for affine dynamics

Consider the N-person affine quadratic game, described by the state equation

$$x_{k+1} = Ax_k + \sum_{i \in \mathbf{N}} B^i u_k^i + c$$

(where $\mathbf{N}$ is the set of players) with cost function for each player i

$$J^i = \sum_{k=1}^{K} c_k^i(x_{k+1}, u_k^i), \qquad c_k^i(x_{k+1}, u_k^i) = x_{k+1}^\top Q^i x_{k+1} + {u_k^i}^\top R^i u_k^i.$$

with $x_k, c \in \mathbb{R}^n$, $u_k^i \in \mathbb{R}^{d_i}$, $A \in \mathbb{R}^{n\times n}$, $B^i \in \mathbb{R}^{n\times d_i}$, $Q^i \in \mathbb{R}^{n\times n}$, and $R^i \in \mathbb{R}^{d_i \times d_i}$. Assume that the matrices have been chosen such that $Q^i \succeq 0$ and $R^i \succ 0$.

a) Consider the last stage K, and write the first-order necessary conditions for the optimal feedback strategies that minimise the remaining cost-to-go $c_K^i(x_{K+1}, u_K^i)$.

b) Prove that the Nash equilibrium strategies at time K must be affine in x_K, i.e. they must have the form

$$\gamma_K^{i*} = -P_K^i x_K - \alpha_K^i \ , \ i \in \mathbf{N}$$

and explain in detail how you can compute the parameters P_K^i and α_K^i.

c) Show that the value function at time K is a function of the state x of the form

$$V_K^i = x_K^\top S_K^i x_K + (r_K^i)^\top x_K + q_K^i \ , \ i \in \mathbf{N}.$$

d) Using an induction argument, provide recursive expression for the computation of the sub-game perfect Nash equilibrium strategy for the entire game (stages $1, \dots, K$).

e) Consider a scalar LQ game with two agents, where $n = 1$ and $d_i = 1$ $\forall i \in \{1, 2\}$. Is the Nash equilibrium unique?

f) Provide an algorithm to compute a Nash equilibrium strategy for this game which is NOT a sub-game perfect Nash equilibrium.

Problem 2. A three-truck platoon

Consider the three-truck platoon example described at slide 12 of the Dynamic games lecture (Lecture 8) as an LQ game. The leading truck is moving at a constant speed, $s^{(1)}$. The second truck aims to maintain a distance, D, both from the first truck ahead and the third truck behind. Similarly, the last truck aims to keep the same distance D from the second truck. Consider $x^{(i)}$ and $s^{(i)}$ as respectively the positions and speeds of trucks, for $i = 1, 2, 3$. The derivative of the position is equal to the speed

$$\dot{x}^{(i)} = s^{(i)},$$

and the derivative of the speed is equal to the acceleration

$$\dot{s}^{(i)} = u^{(i)}.$$

Using Euler discretization, the dynamics can then be written as

$$x_{k+1}^{(i)} = x_k^{(i)} + \Delta_T s_k^{(i)}$$
$$s_{k+1}^{(i)} = s_k^{(i)} + \Delta_T u_k^{(i)}$$

where Δ_T is the discretization time.

Let the relative position of trucks i, j be denoted by $d_k^{(ij)} = x_k^{(i)} - x_k^{(j)}$.

a) Write the discrete-time equation for the relative positions.

b) Identify the state matrix A, the control matrices B_i, and the cost matrices Q_i and R_i.

c) Write the equations corresponding to the subgame perfect Nash equilibrium.

Hint: the quadratic term in the cost function will not be $d^\top Q d$ *but* $(d - \bar{d})^\top Q (d - \bar{d})$ *. For this reason, the value function of the an agent is given by three terms:* $V_{2,k}(d) = d^\top P_{2,k} d + (r_{2,k})^\top d + C_{2,k}$*. Thus, for the second question of the problem, you can repeat the steps at slide 26 of Lecture 8 with the value function proposed. When an equation becomes too long, substitute some terms with new variables to simplify the writing.*